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Estimating Reconstruction Error due to Jitter of Gaussian Markov Processes

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***Abstract*—This paper presents estimation of reconstruction error due to jitter of Gaussian Markov Processes. Two samples are considered for the analysis in two different situations. In one situation, the first sample does not have jitter while the other one is effected by jitter. In the second situation, both the samples are effected by jitter. The probability density functions of the jitter are given by Uniform Distribution and Erlang Distribution. Statistical averaging is applied to conditional expectation of random variable of jitter. From that, conditional variance is obtained which is defined as reconstruction error function and by knowing that, the reconstruction error of a Gaussian Markov Process is determined.**

Keywords—Jitter, Uniform Distribution, Erlang Distribution, Gaussian Markov Process, Probability Density Function.

I. Introduction

A Gaussian Markov process is the process which satisfies the requirements of both Gaussian and Markov processes. These are the stochastic processes that are desired to be recovered. While variation in the precise location of sampling instants T_n is referred to as Jitter. Thus the sampling instants become random too.

The work on the reconstruction problem of a function was first done in the year 1957 by A.V. Balakrishnan [1] in which he tried to reconstruct a Wide Sense Stationary process. Later on in the year 1962, the same author worked on the reconstruction of random processes with jitter [2]. The same work was then continued by Brown [3].

Later on, the work of Balakrishnan and Brown were generalized by B. Liu and T. P. Stanley. Their work includes calculation of error bounds introduced because of jitter in the sampling stage.

Most of the researchers in their articles like W. M. Brown [3], B. Liu et.al [4], M. Shinagawa et.al [5], Kurosawa et.al [6], A. Tarczynski et.al [7], C. Feng et.al [8], B. Atakan [9], E. Van der Onderaa and J. Renneboog [10], P. Marziliano and M. Vatterli [11], N. Da Dalt, et.al [12] have claimed that the best way to reconstruct from samples in the presence of jitter in sampling instants is the way so defined in [2]. In all of the stated research articles, the signal was defined to be real and stationary in the wide sense and its power spectral density (PSD) disappears outside the range of angular frequency ($-\omega_b$, ω_b) whereas $\omega_b = 2\pi W$.

There are some principal drawbacks of majority mentioned publications:

1) The number of samples is equal to infinity,
2) The information about the probability density function (pdf) of the sampled process is not used,
3) All samples have the same jitter distribution.

In this article Conditional Expectation is referred to as ideal algorithm for the reconstruction of Gaussian processes. With this approach being applied, new aspects of the problem are investigated:

1) The number of samples are arbitrary.
2) The covariance function of the random process is taken into use, without any concern of its power spectral density being bandlimited or not.
3) The reconstruction error function is represented on the whole time domain, thus a greater detail of the reconstruction error of a process between any sampling intervals is obtained.

II. Proposed Methodology

Let a non-stationary random process x(t) which is discretized in time instants $\tau = (\tau_1, \tau_2, \ldots, \tau_N)$. Thus we obtain a multiple samples of the process $x(\tau)$, where the number of samples N and sampling instants τ are arbitrary. And so we get a new random process having conditional probability density function f(x), and central moments depending upon each sample value.

$$f(x(t)|X,\tau) = f(x(t)|x(\tau_1), x(\tau_2), \ldots, x(\tau_N)), \qquad (1)$$

In case of arbitrary pdfs, there exist a statistical approach referred to as conditional expectation which promises minimum error of estimate. Following this approach, the conditional expectation also known as the conditional mean function is used as reconstruction function while conditional variance function is used as reconstruction error function. In case of Gaussian processes, the covariance function $K(\tau_i, \tau_j)$ is widely used. The conditional expectation and variance of a non-stationary Gaussian process are given below:

$$\tilde{m}(t) = m(t) + \sum_{i=1}^{N}\sum_{j=1}^{N} K_x(t, \tau_i) a_{ij}\left[x(\tau_i) - m(\tau_j)\right] \quad (2)$$

$$\tilde{\sigma}^2(t) = \sigma(t) - \sum_{i=1}^{N}\sum_{j=1}^{N} K_x(t, \tau_i). a_{ij} K_x(\tau_j, t) \quad (3)$$

Whereas the term a_{ij} represents each term of the inverse covariance matrix:

$$A = \begin{bmatrix} K_x(\tau_1, \tau_1) & \cdots & K_x(\tau_1, \tau_N) \\ \vdots & \ddots & \vdots \\ K_x(\tau_N, \tau_1) & \cdots & K_x(\tau_N, \tau_N) \end{bmatrix}^{-1} \quad (4)$$

The expressions (2) and (3) depend on the current number of sample j, on the total number of samples N, on the set of arbitrary sampling instants τ_i, on the covariance moment among process sections at the instants τ_i and τ_j and on the covariance moment $K_x(t, \tau_i)$ among the current sections of the time t.

Conditional expectation is used directly in case of known or definite sampling instants. But in case of jitter, when the sampling instants are not known, but random then statistical average with respect to the pdf of jitter is applied to the above stated rule in order to estimate the conditional functions.

With the help of this rule, and having the number of samples N finite, the reconstruction of a random process inside the sampling region and outside the sampling region can be described.

III. JITTER

In this paper the sampling instants i.e.$\tilde{\tau}$ are random because of presence of jitter. Also the jitter present in each sample isn't necessary to be the same as it can variate from sample to sample. Thus the pdf of jitter in each sample differ from each other. The pdfs of the jitter considered in this article are uniform or rectangular distribution as shown in figure 1,

$$f(\tilde{\tau}) = \begin{cases} 0\,, \ \tilde{\tau} < a \\ \frac{1}{b-a}\,, a \le \tilde{\tau} \le b, \\ 0, \ \tilde{\tau} \ge b \end{cases} \quad (5)$$

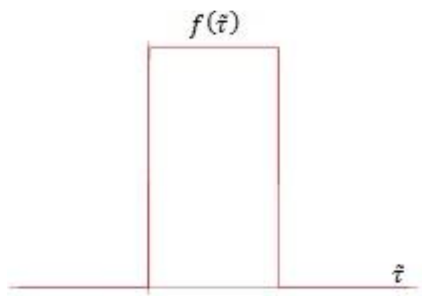

Figure 1. pdf of Uniform Distribution

and the Erlang distribution [13]. Erlang Distribution was developed by A. K. Erlang to examine the number of telephone calls which might be made at the same time to the operators of the switching stations. This work on telephone traffic engineering has been expanded to consider waiting times in queueing systems in general. The distribution is now used in the fields of stochastic processes and of biomathematics. It can be used to model the time to complete n operations in series, where each operation requires an exponential period of time to complete. The pdf of Erlang distribution is shown in figure 2 and its mathematical form is given below:

$$f(\tilde{\tau}) = \lambda \,.\, e^{-\lambda\tilde{\tau}} \,.\, \frac{(\lambda\tilde{\tau})^{k-1}}{(k-1)!} \quad ,\lambda, \tilde{\tau} \ge 0 \quad (6)$$

Erlang distribution has two parameters as seen in the pdf, k and λ. The k parameter is referred to as the shape parameter while the λ is referred to as the rate parameter. When the value of shape parameter k equals 1, it takes the form of exponential distribution.

An alternative, but equivalent, parametrization uses the scale parameter μ which is the reciprocal of the rate parameter (i.e. μ =1/ λ).

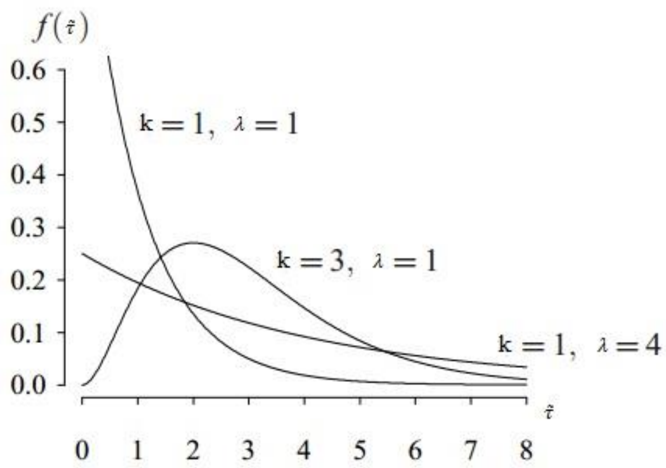

Figure 2. pdf of Erlang Distribution

Two different cases are considered. 1) When the first sample is definite while the other sample has jitter i.e. the position of the second sample is random $\tilde{\tau}$. 2) When both the samples have jitter i.e. both the sampling instants are random variables.

For the first case, the average reconstruction function which is the average conditional expectation depends upon the pdf of the jitter present in the second sample $f(\tilde{\tau})$, so we get:

$$\langle \tilde{\sigma}^2(t) \rangle = \iint \tilde{\sigma}^2(t). f(\tilde{\tau}_2) d\tilde{\tau}_2 \quad (7)$$

For the second case, the average conditional expectation depends upon the pdfs of jitter of both samples i.e. $f(\tilde{\tau}_1), f(\tilde{\tau}_2)$. Thus the average reconstruction error is calculated as follows:

$$\langle \tilde{\sigma}^2(t) \rangle = \iint \tilde{\sigma}^2(t). f(\tilde{\tau}_1)\, f(\tilde{\tau}_2)\, d\tilde{\tau}_1\, d\tilde{\tau}_2 \quad (8)$$

In this article, the random process used throughout is obtained at the output of a low pass RC filter with white noise as the input.

Thus a Gaussian Markov process well defined by the covariance function given below is obtained:

$$K(\tau_i, \tau_j) = \sigma^2 . e^{-\alpha|\tau_i, \tau_j|} \quad (9)$$

Whereas α=1/RC. In the said filter, the value of α is assumed to be 1, so that the covariance time has a value 1.

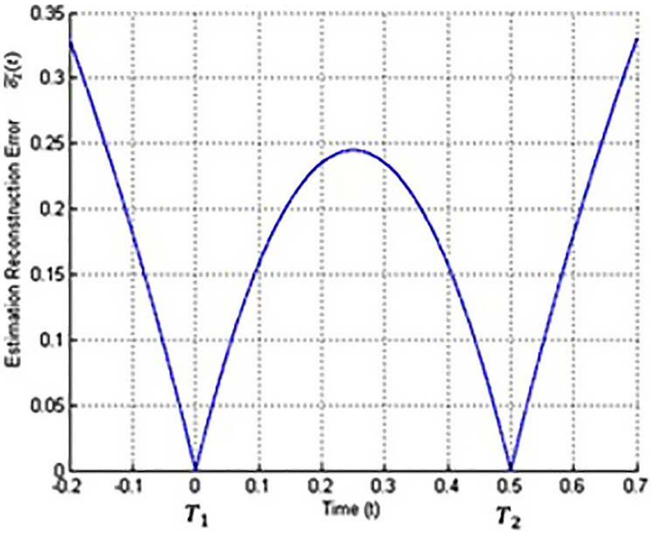

Figure 3. Reconstruction Error Function with both samples having no Jitter.

IV. Examples

The conditional expectation also referred to as conditional mean defines the conditional variance which is defined as the reconstruction error function.

Here, in all the examples considered, the two sampling instants taken are at 0 and 0.5. While the covariance time used is 1.

The reconstruction error function without any jitter as defined by (3) is represented in the figure 3. From the figure it is visible that the function is defined on the whole time domain. Also it can be seen that the reconstruction error at sampling points have value 0 as at these points the value of the process is exactly known. While exactly in the center of both the sampling intervals, the value of the reconstruction error is maximum which is obvious as there is greater uncertainty about the exact value of the process at this instant.

On the basis of (7), (8) a set of examples are presented to clearly show the effect of jitter during reconstruction. These expressions describe the average reconstruction function with random sampling intervals.

The multiple scenarios we have considered are as follows:

- One sample with random sampling intervals.
  - The jitter of the sample described by uniform pdf.
  - The jitter of the sample described by Erlang Distribution function.
- Both the samples with random sampling intervals with jitter defined by Uniform pdf.
  - Both samples having same jitter.
  - Both samples with different jitter.

A. *When the second sample has random sampling instant i.e it has jitter which is defined by Uniform Probability Distribution*

In this case, the process is carried out with the help of average reconstruction error function (7). The second sampling instant is effected by jitter represented by ε_2, thus the sampling instant becomes:

$$\tilde{\tau}_2 = \tau_2 + \varepsilon_2 \quad (10)$$

The random variable ε_2 in this case is defined by Uniform Probability Density Function and is limited between the interval of 0.05seconds on either side of the sampling point. So $\tilde{\tau}_2$ becomes as shown in (11):

$$[\tilde{\tau}_2 - 0.05, \tilde{\tau}_2 + 0.05] = [0.45, 0.55] \quad (11)$$

The reconstruction error function is shown in figure 4. In the figure it can be seen that the reconstruction estimate is erroneous only in the region where there is effect of jitter i.e. between 0.45 and 0.55. As compared with the original samples without jitter, the jitter in here causes 5% error in the sample.

B. *When the jitter of the second sample is defined using Erlag Distribution*

In this scenario, the jitter in the second sample is described using Erlang Distribution.

For instance four different distributions are used having the following values.

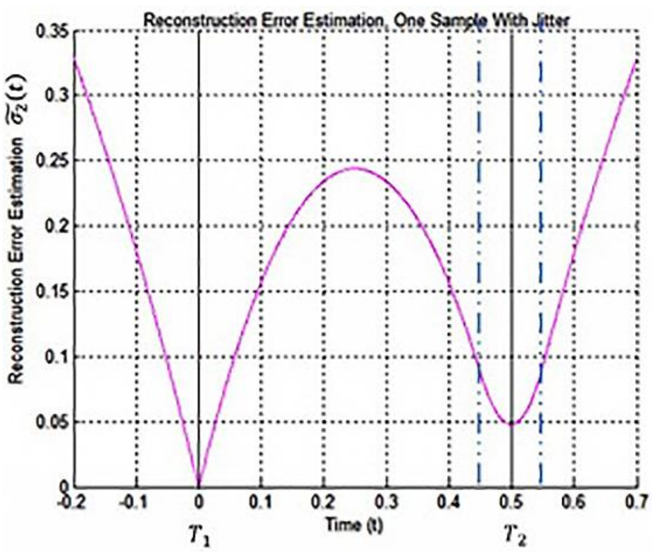

Figure 4. The reconstruction error function when one sample is without jitter and the other sample have the presence of jitter defined by Uniform pdf.

$$k = 1, \lambda = 85$$
$$k = 2, \lambda = 85$$
$$k = 2, \lambda = 65$$
$$k = 2, \lambda = 45$$

The pdfs of the jitter is given in figure 5. The reconstruction error function is shown in figure 06. Where it is clearly visible that the maximum error between all the four different samples is different in both values and position.

It is because of the difference between average distances of the samples. Also, with the increase in this difference, the maximum error which lies exactly in middle of the two sampling points where the uncertainty is maximum increases. Also, from the figure, it is clear that the error is effecting the value at the sampling instant too with the variance of the jitter.

C. *When both the samples have same jitter defined by Uniform Distribution Function*

In this case, the jitter is present in both the samples, defined by Uniform Distribution. The jitter is exactly the same. In order to clarify the case in a more better form, three different jitter widths are taken with the following limits.

- [-0.025, 0.025] i.e. total width = 0.05.
- [-0.05, 0.05] i.e. total width = 0.1.
- [-0.1, 0.1] i.e. total width = 0.2.

These widths are represented using blue, green and red dotted line in figure 7.
Figure 7 gives reconstruction error function for each of the jitter width in both samples. From this example, it is clear that the error increases with the increase in jitter width as it makes the exact value of sample at the sampling instant more uncertain.

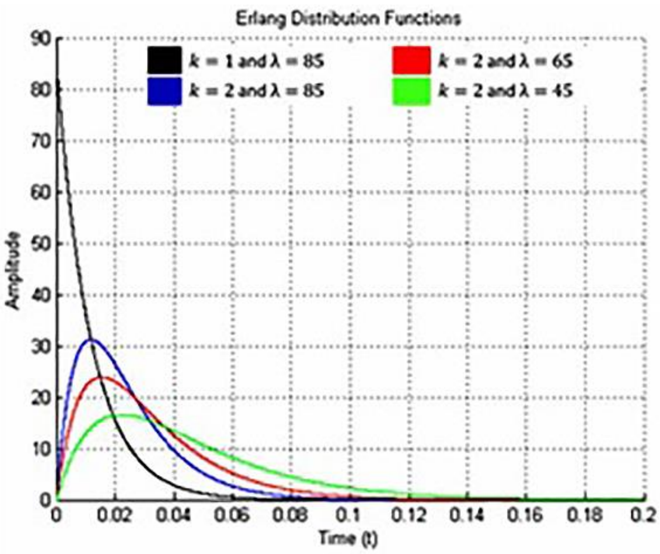

Figure 5. Four different Jitters represented by Erlang distribution of different parameters

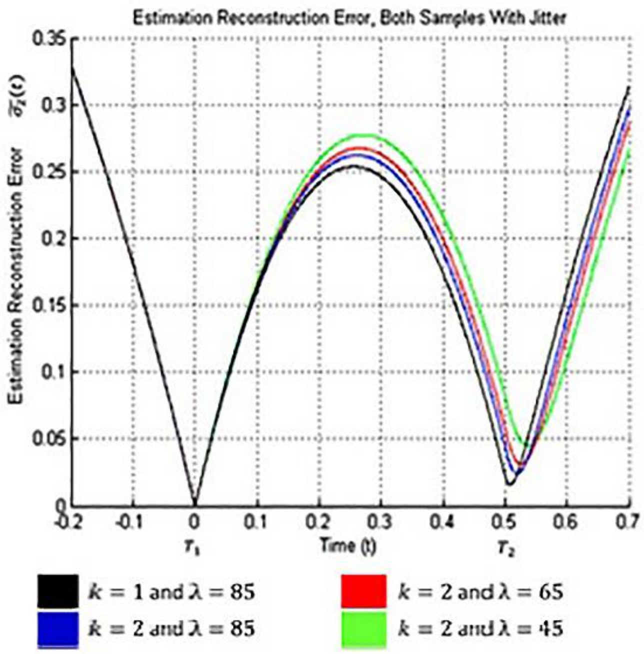

Figure 6. Reconstruction Error Function when one sample has jitter defined by Erlang distributions of different parameters as shown in figure 5.

D. *When both the samples have jitter of different widths defined by Unifrom Distribution Function*

And lastly the case when both the samples have different jitter widths, well described by Uniform Probability Density Function is studied. The jitter width of the first sample is [-0.025, 0.025] with total width of 0.5 while the width of the jitter of the other sample is [-01, 0.1] of total width 0.2.

The reconstruction error function is given in figure 8. Where it can be seen that error in the sample with smaller jitter width is small as compared to the other one of larger jitter width.

From this example, it can be concluded that the characterization of jitter of samples with different distributions is also possible.

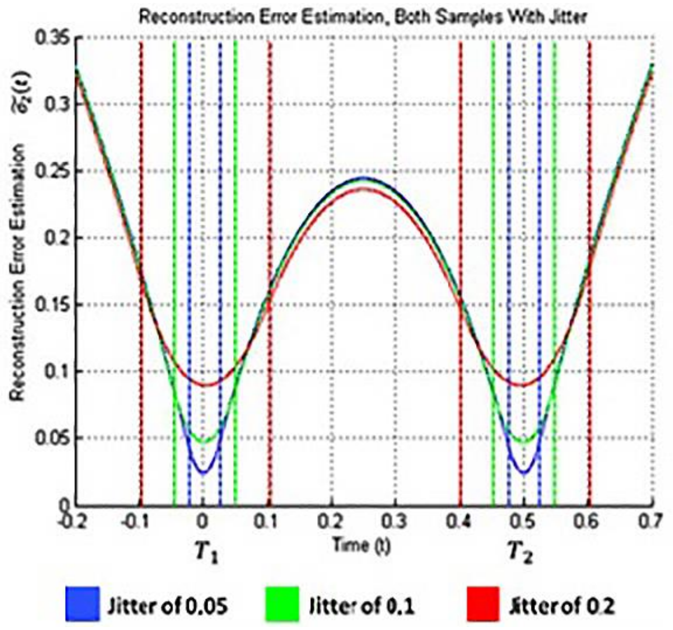

Figure 7. Reconstruction Error Function when both the samples have jitter of same width at a time. This figure depicts three different cases at a time.

V. Conclusion

From this paper, it can be concluded that the conditional expectation or conditional mean rule permit us to depict the sampling-reconstruction process for Gaussian Markov processes where there is or not the presence of jitter in the sampling instants. With this, it is conceivable to study that how the reconstruction of the signal from its samples is influenced on the whole time domain in presence of jitter. It was shown that it is conceivable to bring the investigation under different distinctive cases, for instance: when there is no jitter; when just a single sample has jitter; when both samples have jitter with the same or different pdf of different distribution functions. Shortly, the reconstruction error increases with the increase in jitter.

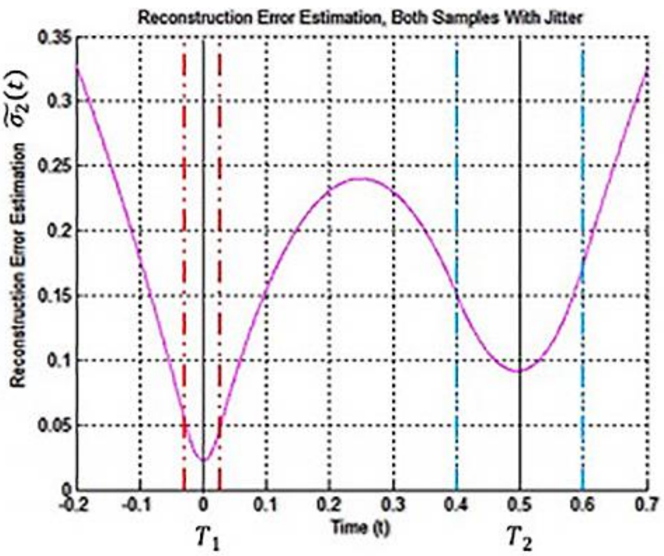

Figure 8. Reconstruction Error Function when both the samples have jitter of different widths represented by Uniform Probability Density Function.

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